A Non-Oscillatory Central Scheme for Conservation Laws

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ABSTRACT

A non-oscillatory central scheme, originally developed by Nessyahu and Tadmor[1], has been modified based on a strategy of space-time integration similar to that in the Space-Time Conservation Element and Solution Element Method, originally developed by Chang[2]. The resulting scheme is a two-point, second-order, central difference scheme, which retains most of the advantageous features of Chang's space-time method. Similar to both Nessyahu and Tadmor's method and Chang's CE/SE method, the present method does not use the Riemann solver as the building block as that in the modern upwind schemes. Thus the logic is considerably simpler and the computation is much more efficient. In addition, based on space-time flux balance, the boundary condition treatment in the present method is easier than Nessyahu and Tadmor's method. To demonstrate the capabilities of the present scheme, numerical results of some standard problems are reported, including the Sod shock tube problem, the Shu-Osher problem of a shock interacting with a sinusoidal wave, the Woodward-Colella problem of two interacting blast waves, and the oblique shock reflection problem. The results show that the present scheme is accurate and efficient for moving shocks and wave motions.

1 INTRODUCTION

Based on the first-order Lax-Friedrichs (LxF) scheme, a non-oscillatory second-order central scheme for hyperbolic conservation laws is developed by Nessyahu and Tadmor. We refer to the method as the NT scheme hereafter. In the NT scheme, a second-order piecewise-linear function was employed instead of the first-order piecewise constant method used in the LxF scheme to eliminate excessive artificial damping of the LxF scheme. As a second-order central difference scheme, the main advantage the NT scheme is its simplicity in shock capturing. In particular, no Riemann solver is employed. Recently, in a series of works, the NT scheme has been extended to be third-order scheme[3], and has also been used to solve flows in multiple spatial dimensions [4]. In [5], the NT method has been extended to solve incompressible flows.

From a totally different prospective, Chang [2] proposed the Space-Time Conservation Element and Solution Element method, or the CE/SE method for short. The method has many non-traditional features, including simplicity, generality, high accuracy and high shock resolution for moving shock and wave motions. The method also has the most compact stencil among the usual numerical methods. Essentially, both flow variables and their gradient components were treated as unknowns and they were solved simultaneously. Triangles and tetrahedrons were used for two and three-dimensional flows, respectively. This space-time mesh arrangement is in full compliance with the flow physics. In particular, the a-scheme of the CE/SE family is space-time invariant and it is a non-dissipative scheme for continuous solution.

Recently, based on Chang's CE/SE method, we have developed a modified space-time method using quadrilateral mesh [6]. The modified CE/SE method retains most of the advantageous features of the original

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CE/SE method. By comparing the NT scheme with the modified CE/SE method, we observed significant similarities. The objective of the present paper is to discuss relationship between the NT scheme and the modified CE/SE method. Based on this underpinning relationship, we then propose a variation of the NT scheme based on the concept of the space-time flux balance.

In the present paper, we first discuss an alternative construction of the NT scheme. With that we can clarify the relation between the NT scheme and the modified CE/SE scheme [6]. In section 3, the original NT scheme is modified using the strategy similar to that in the CE/SE method. Essentially, the numerical derivatives are reconstructed from flow variables of neighboring points at the previous time level. This is in full compliance with the flow physics of the initial value problems. We remark that, in the original NT scheme, the construction is based on flow variables at the new time level. In addition, the NT scheme in one spatial dimension is actually a four-point scheme. Thus it is relatively complex for implementation near boundaries. Because of this reason, the accuracy of the NT scheme deteriorates near the computational boundaries.

The result of above modification is a two-point second-order central scheme, which is easier to be implemented at mesh nodes near boundaries, and the accuracy of the new scheme does not deteriorate near the boundaries. This scheme retains the advantages of the original NT scheme and the CE/SE method.

The remainder of this paper is organized as follows. In Section 2, an alternative construction of the original NT scheme is discussed. Based on this new construction, the present scheme is proposed in Section 3. In section 4, numerical results of some standard flow problems are presented to show capabilities of the present scheme.

2 An Alternative Construction of the NT Scheme

In this section, we provide an alternative construction of the NT scheme based on the space-time integration of Chang's CE/SE method. To proceed, we consider the Euler equations of a perfect gas in one spatial dimension

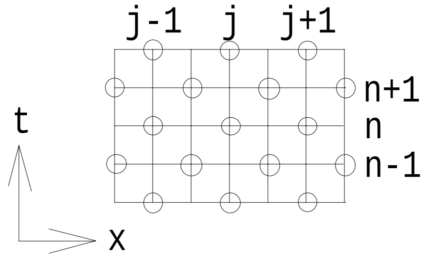
$$\partial u_m / \partial t + \partial f_m / \partial x = 0\,, \quad m=1,2,3 \qquad (2.1)$$

Let $x_1=x$ and $x_2=t$ be two coordinates of an Euclidean space E_2. By Gauss' divergence theorem in E_2, Eq.(2.1) has the following integral counterpart:

$$\oint_{S(V)} \vec{h}_m \cdot d\vec{s} = 0\,, \quad m=1,2,3 \qquad (2.2)$$

Where S(V) is the boundary of an arbitrary space-time region V in E_2, and $\vec{h}_m = (f_m, u_m)$ is the space-time current density vector.

To proceed, we illustrate the definitions of conservation element (CE) and solution element (SE), which is of importance in the formulation of CE/SE scheme. A CE is a small region in E_2 in which Eq.(2.2) is enforced. While, a SE is a region in E_2 in which flow variables are discretized by chosen functions, e.g., the piecewise linear function in Chang's CE/SE method. Following the CE/SE method, the representative mesh points distribution in E_2 is shown in Fig.2.1(a), which is staggering in time. Let Ω denotes the set of mesh points in E_2 (dots in Fig.2.1(a)). For each point (j, n)∈ Ω, there is a SE and a CE associated with it. For example, at point A, the CE is defined as the quadrilateral BCEF, while SE is the interior of the quadrilateral BDFG (see Fig.2.1(b))



(a)

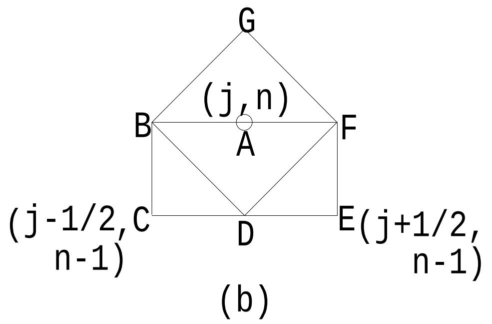


Fig.2.1 The representative mesh distribution and the definitions of SE and CE

By the definition of SE, $\forall$ (x, t)$\in$SE(j, n), $u_m(x, t)$ can be approximated by its discrete counterpart $u_m^*(x, t; j, n)$. Here, we use the following first-order Taylor expansion:

$$u_m^*(x,t;j,n) = (u_m)_j^n + (u_{mx})_j^n (x - x_j) + (u_{mt})_j^n (t - t^n) \tag{2.3}$$

Correspondingly, we have

$$f_m^*(x,t;j,n) = f(u_m^*(x,t,j,n)) \tag{2.4}$$

$$h_m^*(x,t;j,n) = (f_m^*(x,t,j,n), u_m^*(x,t,j,n)) \tag{2.5}$$

Then, Eq.(2.2) can be approximated by the following discrete one:

$$\oint_{S(CE(j,n))} \vec{h}_m^* \cdot d\vec{s} = 0 \;, \qquad \forall(j, n)\in\Omega \tag{2.6}$$

On the other hand, by Eq.(2.1), we have

$$u_{mt} = -A(u_m)\, u_{mx} \tag{2.7}$$

Here $A = \partial f / \partial u$ is the Jacobian matrix of the flux f_m. So it can be shown that the only independent discrete variables needed to be solved in the current marching scheme at each grid point are u_m and u_{mx}.

Substituting Eqs.(2.3-5) into Eq.(2.6), one can get:

$$\begin{aligned}(u_m)_j^{n+1} &= [(u_m)_{j-1/2}^n + (u_m)_{j+1/2}^n]/2 \\ &+ [(u_{mx})_{j-1/2}^n - (u_{mx})_{j+1/2}^n]\cdot \Delta x/8 \\ &+ \int_n^{n+1} f(u_m^*(x_{j-1/2}, t; j-1/2, n))dt \\ &- \int_n^{n+1} f(u_m^*(x_{j+1/2}, t; j+1/2, n))dt \end{aligned} \tag{2.8}$$

If the last two terms on the right of Eq.(2.8) are integrated approximately by the midpoint rule, we have

$$\begin{aligned}(u_m)_j^{n+1} &= [(u_m)_{j-1/2}^n + (u_m)_{j+1/2}^n]/2 \\ &+ [(u_{mx})_{j-1/2}^n - (u_{mx})_{j+1/2}^n]\cdot \Delta x/8 \\ &+ \lambda \cdot [f((u_m)_{j-1/2}^{n+1/2}) - f((u_m)_{j+1/2}^{n+1/2})]\end{aligned} \tag{2.9}$$

Here $\lambda=\Delta t/\Delta x$, $\Delta x=x_{j+1/2}-x_{j-1/2}$ and $\Delta t=t^{n+1}-t^n$. And

$$(u_m)_{j\pm1/2}^{n+1/2} = (u_m^*(x, t;\, j\pm1/2, n))\Big|_{x_{j\pm1/2}}^{t^{n+1/2}}$$

According to Eq.(2.3), one can get

$$(u_m)_{j\pm1/2}^{n+1/2} = (u_m)_{j\pm1/2}^n + (\Delta t/2)(u_{mt})_{j\pm1/2}^n \tag{2.10}$$

Eqs.(2.9-10) is just the NT scheme (see (2.13) and (2.14) in ref.[1]). Here, the NT scheme is derived by using another more natural method, which is similar to that of the CE/SE method. In fact, if we let

$$f((u_m)_{j\pm1/2}^{n+1/2}) \equiv A((u_m)_{j\pm1/2}^{n+1/2})\cdot(u_m)_{j\pm1/2}^{n+1/2} \cong A((u_m)_{j\pm1/2}^{n})\cdot(u_m)_{j\pm1/2}^{n+1/2}, \tag{2.11}$$

Then Eq.(2.9) is the same as the main discrete equation of the CE/SE method (see refs. [2, 6]).

3 MODIFIED NT SCHEME

As mentioned in ref.[1], the resolution of NT scheme hinges upon the choice of local numerical derivatives u_{mx} and f_{mx}. The following formulations are used in [1]:

$$(u_{mx})_j = MM\{\Delta(u_m)_{j-1/2}, \Delta(u_m)_{j+1/2}\}/\Delta x \tag{3.1}$$

or

$$(u_{mx})_j = MM\{2\Delta(u_m)_{j-1/2}, [(u_m)_{j+1} - (u_m)_{j-1}]/2, 2\Delta(u_m)_{j+1/2}\}/\Delta x \tag{3.2}$$

and

$$(f_{mx})_j = A((u_m)_j)(u_{mx})_j \tag{3.3}$$

Here $\Delta(u_m)_{j+1/2} = (u_m)_{j+1} - (u_m)_j$, and MM{..} stands for the usual limiter, i.e.

$$MM\{x, y\} = \min\bmod\{x, y\} = [sgn(x) + sgn(y)]/2 \cdot \min(|x|, |y|)$$

Equations.(2.9-10) with (3.1) is referred to as the STG method, while Eqs.(2.9-10) with (3.2) is referred to as STG2 method (see ref.[1]). If the CFL condition is enforced, it can be shown that STG and STG2 are second-order TVD schemes. We remark that, from Eqs.(3.1) and (3.2), it can be seen that STG or STG2 is actually four-point scheme, so it is not much convenient to implement for the mesh nodes near boundaries. In addition, the accuracy of these schemes will decrease in these mesh nodes.

Instead of using Eq.(3.1) or (3.2), we use the following central difference reconstruction procedure to calculate the local spatial numerical derivative of the flow variables, i.e., u_{mx},

$$(u_{mx})_j^n = [(u_{mx}^+)_j^n + (u_{mx}^-)_j^n]/2 \tag{3.4}$$

Where

$$(u_{mx}^{\pm})_j^n = \pm[(u_m')_{j\pm1/2}^n - (u_m)_j^n]/(\Delta x/2)$$

$$(u_m')_{j\pm1/2}^n = (u_m)_{j\pm1/2}^n + \Delta t \cdot (u_{mt})_{j\pm1/2}^n$$

It should be noted that Eq.(3.4) is the special case of the CE/SE method for ε=0.5, see Eq.(4.28) in ref.[2]). Eqs.(2.9-10) with (3.4) is the modified NT scheme for solving the 1-D Euler equations. It is a real two-point second-order explicit scheme, so it is easier to implement near boundaries, and it is also in full compliance with the flow physics of the initial value problems. For flows with discontinuities, Eq.(3.4) can be further modified by the following re-weighting procedure:

$$(u_{mx})_j^n = W((u_{mx}^-)_j^n, (u_{mx}^+)_j^n, \alpha) \tag{3.5}$$

or

$$(u_{mx})_j^n = MM\{(u_{mx}^-)_j^n, (u_{mx}^+)_j^n\} \tag{3.6}$$

Where W is the re-weighting function defined by

$$W(x_-, x_+, \alpha) = \begin{cases} 0, & if \ x_- = x_+ = 0 \\ \dfrac{|x_+|^\alpha x_- + |x_-|^\alpha x_+}{|x_+|^\alpha + |x_-|^\alpha}, & otherwise \end{cases}$$

And α is an adjustable constant, usually α=1 or 2.

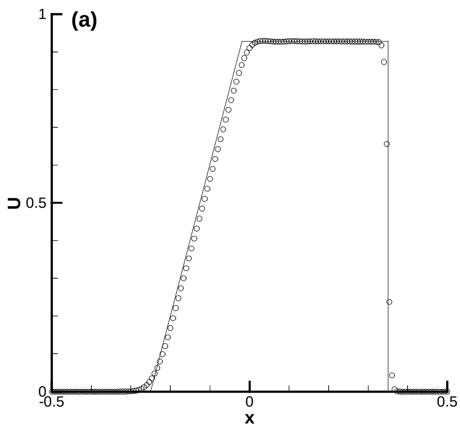
The above scheme can be extended to multi-dimensional cases in a straightforward manner. The details about this extension will be reported in another paper. Nevertheless, a numerical example of a two-dimensional flow problem is provided in Section 4.

4 Numerical Examples

In order to compare the solution accuracy and shock resolution of the original NT scheme and the modified one, four prototype flow problems are calculated using these schemes.

Example 1 Sod's shock tube problem [7]

The computational domain and the flow condition are the same as that in Ref. [1]. Figure 4.1 shows the numerical results at t = 0.2, calculated by STG, STG2 and modified NT schemes, here $\Delta t = 0.1\times10^{-2}$, CFL ≅ 0.6.



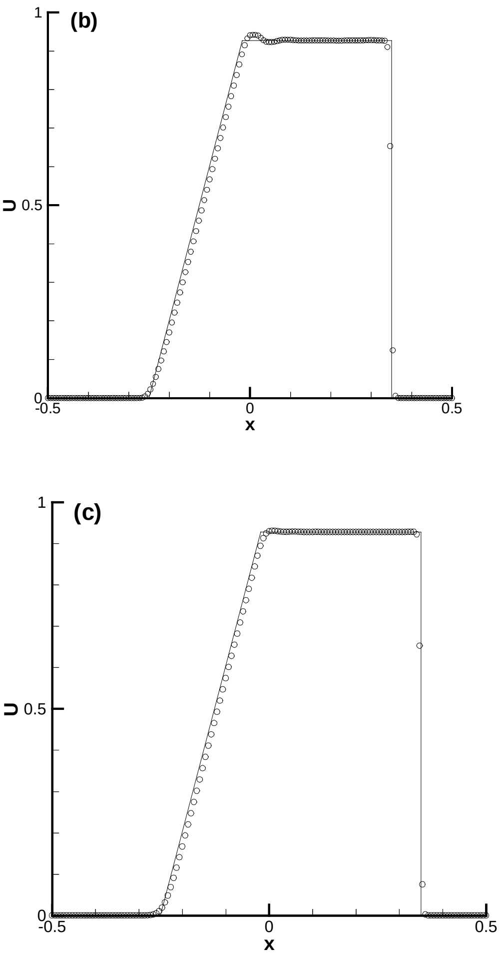


Fig.4.1 The numerical solution of Sod's problem

(a—STG; b — STG2; c — Modified NT)

It can be seen that the solution accuracy and shock resolution of STG is not so good as that of modified NT scheme, while the over-shoots of the STG2's solution near the contact discontinue point is more serious than that of the modified NT scheme's.

Example 2 Shu-Osher problem [8]

It is the interaction of a moving shock of Mach number =3 with a sinusoidal density wave. This problem does not have the exact solution. The numerical solutions are compared with a fine-mesh solution. In Fig. 4.2, the solid line is the computed results by the CE/SE method using 1000 grid points.

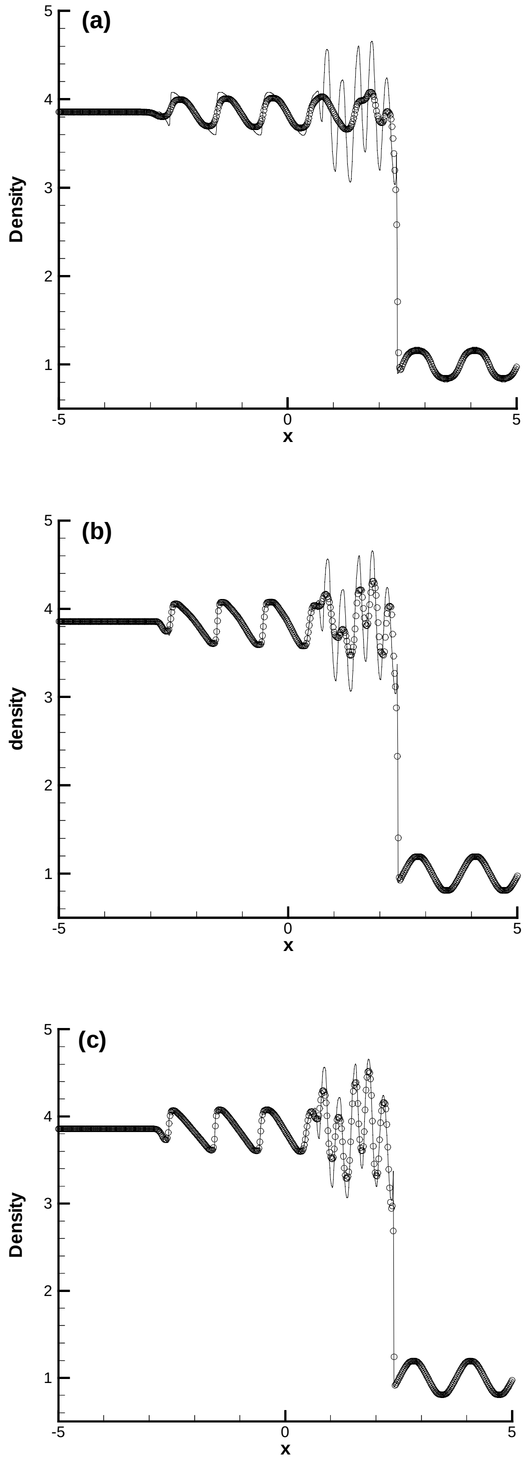


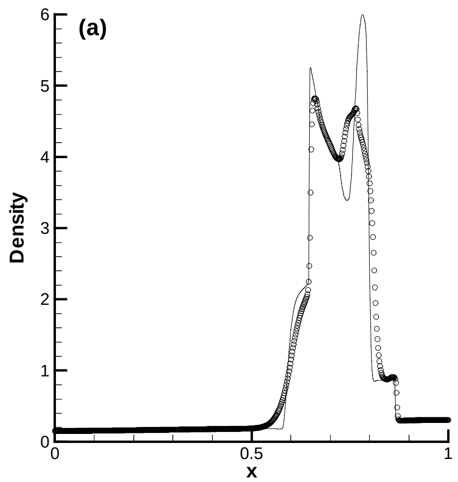
Fig.4.2 The calculated densities of Shu-Osher problem

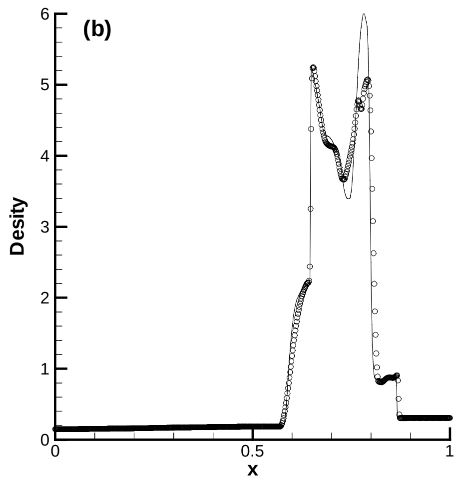
((a)—STG; (b) — STG2; (c) — Modified NT)

Figure 4.2 is the numerical results of STG, STG2 and modified NT schemes with 500 grid points (at t=1.8). The results of the modified NT scheme are better than the STG schemes. With the increase of the grid points, both of the results of STG2 and modified NT will be much better, but the STG's results have no obvious improvement.

<u>**Example 3** The interaction of two blast waves in a tube with closed ends</u>

This problem is proposed by Woodward and Colella[9]. Reflective boundary conditions are imposed at both ends. In Fig. 4.3, we present the results of the STG method, the STG2 method and the modified NT scheme with 600 grid points at t = 0.038. The solid line is the computed results of CE/SE method using 800 grid points. We can see that the modified NT scheme has higher accuracy and its result is better than the other two's.





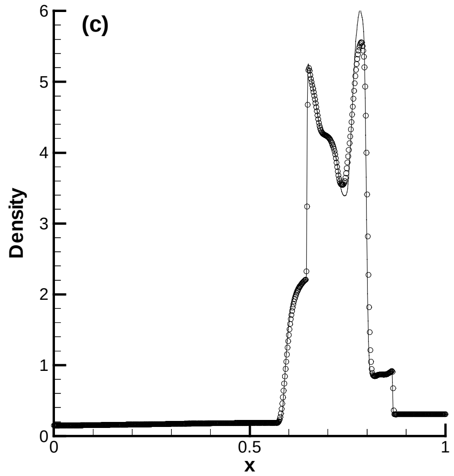


Fig.4.3 The computed densities of Woodward and Colella problem
((a)—STG; (b) — STG2; (c) — Modified NT)

<u>**Example 4** Flat plate shock reflection problem</u>[10]

This is a two-dimensional flow problem, and it has an analytical solution. The computational domain is [0,4]×[0,1]. The lower boundary is a solid wall and a reflective condition is employed. The left lateral and the top horizontal boundaries are fixed according to the analytical solution. While a non-reflective condition is used on the right lateral boundary. Figure 4.4 is the computed pressure contours calculated by the present scheme (121×81 uniform mesh is used). The angle of the reflected shock is very accurate compared with the analytical solution. Although not shown, the numerical result of the pressure jumps agrees well with the exact solution.

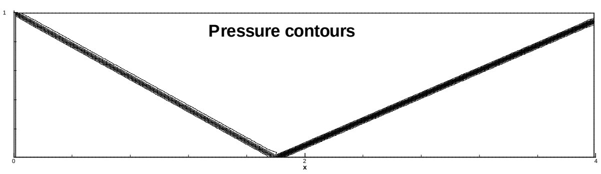


Fig.4.4 Pressure contours of the oblique shock reflective problem

4 CONCLUDING REMARKS

In this paper, the original NT scheme for one dimensional conservation laws is modified using a method similar to the CE/SE method. The present scheme retains most of the advantages of the original NT scheme and the original CE/SE method such as simplicity, efficiency and robustness. Moreover, the present scheme is more accurate and easier to implement near boundaries. Numerical results of several standard problems show that the present scheme has high accuracy and high resolution for shock wave problems. This scheme can be extended to be higher order and for flows in multi-dimensions in a straightforward manner.

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