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Convenient Method to Convert Two-Dimensional CFD Codes into Axisymmetric Ones

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Introduction

ALTHOUGH three-dimensional CFD calculations are commonplace, axisymmetric flow solvers are indispensible tools, especially for flowfields of propulsion systems. In axisymmetric solvers, one applies curvilinear grids in the axial and radial directions and no grid is used in the azimuthal direction. Thus, each control volume is a body of revolution with constant two-dimensional cross-sectional areas in the azimuthal direction. For the time being, we will call these coordinates the axisymmetric coordinate system.

In this note, we propose a convenient and systematic procedure to convert two-dimensional CFD codes into axisymmetric ones. First, we organize the governing equations in a form suitable for CFD applications. Then, we carefully examine the calculations of the volume and surface areas of the axisymmetric control-volume element. Through the conversion process, the procedures of modifying the two-dimensional codes becomes self-evident. Although we concentrate on the finite-volume method in this note, similar procedure could be applied for finite-difference codes.

Governing Equations

The axisymmetric flow equations can be cast into a vector form

$$\frac{\partial y\boldsymbol{Q}}{\partial t} + \frac{\partial y\boldsymbol{E}}{\partial x} + \frac{\partial y\boldsymbol{F}}{\partial y} = \frac{\partial y\boldsymbol{E}_v}{\partial x} + \frac{\partial y\boldsymbol{F}_v}{\partial y} + y\boldsymbol{H} \tag{1}$$

where x and y are the axial and radial coordinates, respectively, $\boldsymbol{Q}$ is the vector of dependent variables, $\boldsymbol{E}$ and $\boldsymbol{F}$ are the convective fluxes

$$Q = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{pmatrix}, \quad E = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(\rho e + p) \end{pmatrix}, \quad F = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(\rho e + p) \end{pmatrix} \tag{2}$$

$\boldsymbol{E}_v$ and $\boldsymbol{F}_v$ are the viscous flux vectors

$$E_v = \begin{pmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ u\tau_{xx} + v\tau_{xy} - q_x \end{pmatrix}, \quad F_v = \begin{pmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \\ u\tau_{yx} + v\tau_{yy} - q_y \end{pmatrix} \tag{3}$$

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$\boldsymbol{H}$ is the source vector

$$H = \begin{pmatrix} 0 \\ 0 \\ (p - \tau_{\theta\theta})/y \\ 0 \end{pmatrix} \tag{4}$$

The specific total energy e, shear stress components τ, and heat flux components q, are given as

$$e = \frac{p}{\rho(\gamma - 1)} + \frac{1}{2}(u^2 + v^2) \tag{5}$$

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\nabla\cdot V$$
$$\tau_{yy} = 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\nabla\cdot V \tag{6}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{7}$$

$$q_x = -k\frac{\partial T}{\partial x}$$
$$q_y = -k\frac{\partial T}{\partial y} \tag{8}$$

where the divergence of the velocity vector is

$$\nabla\cdot V = \frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial(yv)}{\partial y} \tag{9}$$

Due to the axisymmetric coordinate system, the source term $(p - \tau_{\theta\theta})/y$ appears in the radial momentum equation, in which

$$\tau_{\theta\theta} = 2\mu(v/y) - \tfrac{2}{3}\mu\nabla\cdot V \tag{10}$$

As shown in Fig. 1, the source term is actually the net effect of the balancing force in the radial direction due to the normal stresses in the azimuthal direction, and it is an inevitable result of the axisymmetric coordinate system. However, no source term is generated in the energy equation because the velocity in the azimuthal direction is assumed to be zero.

When integrating the governing equations in axisymmetric coordinates, two types of numerical calculations are performed, namely, volumetric integration of time marching and source terms, and surface integration of convective flux terms. In what follows, details of the geometric calculations concerning both volumetric and surface integrations are provided.

Volumetric and Surface Integrations

A typical finite-volume cell inside the calculation domain is shown in Fig. 2. The volume of a single cell can be calculated according to the first theorem of Pappus-Guldinus.[1] The theorem states that the volume of a body of revolution is equal

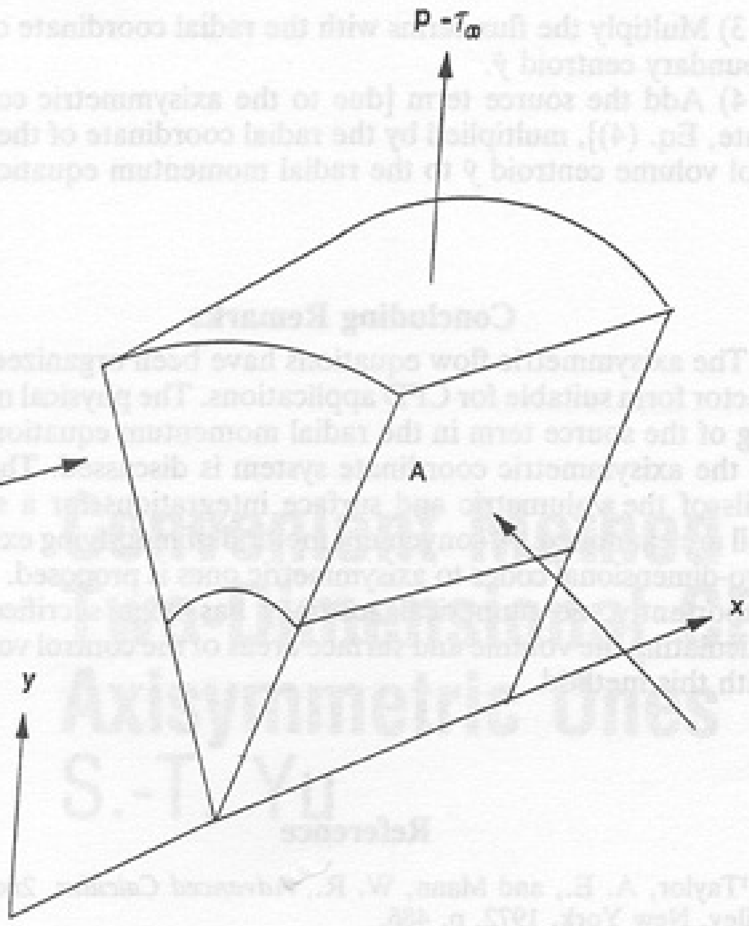

Fig. 1 Source term of the radial momentum equation due to the unbalanced normal stresses in azimuthal direction.

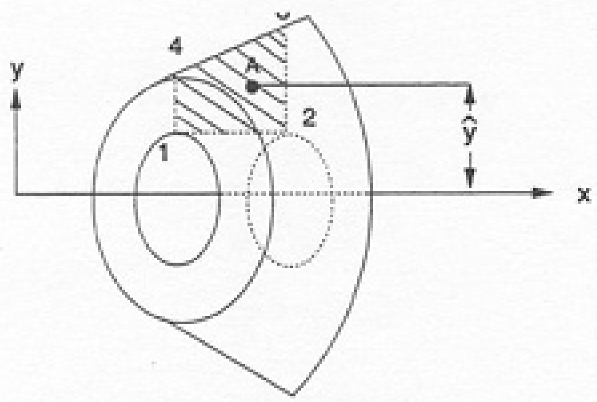

Fig. 2 Control volume of an axisymmetric coordinate system.

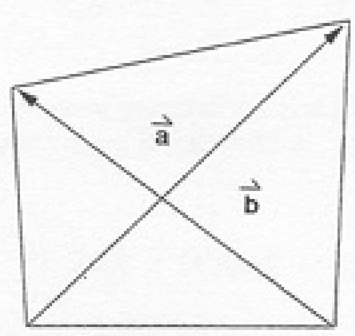

Fig. 3 Calculation of area $A(A = \frac{1}{2}|a \times b|)$.

to the generating area times the distance traveled by the centroid of the area while the body is being generated. Thus

$$V = 2\pi\hat{y}A \tag{11}$$

wherein $\hat{y}$ is the radial coordinate of the centroid of the area A, and can be expressed as $\frac{1}{4}(y_1 + y_2 + y_3 + y_4)$ as shown in Fig. 2. The area A is the intersection of the x-y plane and the axisymmetric control volume; it can be calculated by the cross product of the two diagonal vectors as shown in Fig. 3:

$$A = \tfrac{1}{2}|a \times b| \tag{12}$$

Comparing Eqs. (1) and (11), the volumetric integration of the time-marching term in Eq. (1) over the volume V is equal to the surface integration of the term $\partial(\hat{y}Q)/\partial t$ over the area A multiplied by 2π.

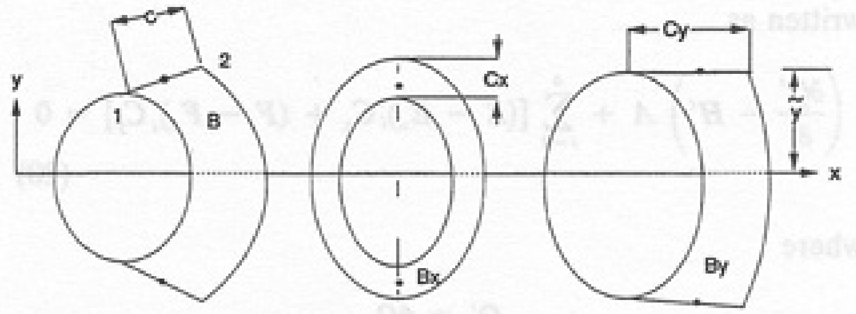

Fig. 4 One of the boundary surfaces of an axisymmetric control volume.

Fluxes need to be calculated on four surfaces of each control-volume cell, therefore, correct calculation of these surface areas is essential. The second theorem of Pappus-Guldinus states that the area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid of the curve while the surface is being generated. Figure 4 shows a typical surface of a control volume. According to the second theorem of Pappus-Guldinus, the surface area is

$$B = \pi(y_1 + y_2)\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{13}$$

Clearly, this surface can be decomposed into two surfaces with the same curve centroid, as shown in Fig. 4.

$$\begin{aligned} B &= B_x + B_y \\ B_x &= \pi(y_1 + y_2)(y_2 - y_1) = 2\pi\bar{y}(y_2 - y_1) \\ B_y &= \pi(y_1 + y_2)(x_2 - x_1) = 2\pi\bar{y}(x_2 - x_1) \end{aligned} \tag{14}$$

where $\bar{y}$ is the radial coordinate of the centroid of surfaces B, B_x, and B_y. Let

$$B = 2\pi\bar{y}C \tag{15}$$

where

$$\begin{aligned} C &= \sqrt{C_x^2 + C_y^2} \\ C_x &= (y_2 - y_1) \\ C_y &= (x_2 - x_1) \end{aligned} \tag{16}$$

In a two-dimensional, curvilinear coordinate system, C_x is actually y_η (or y_ξ) and C_y is x_η (or x_ξ) where $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$.

Let the flow flux vector be defined as

$$G = (E - E_v)x + (F - F_v)y \tag{17}$$

where x and y are the unit vector in axial and radial directions. The net flux that passing through the surface B is

$$\begin{aligned} G\cdot B &= (E - E_v)B_x + (F - F_v)B_y \\ &= 2\pi\bar{y}[(E - E_v)C_x + (F - F_v)C_y] \end{aligned} \tag{18}$$

Therefore, the net flux over the area B is equal to 2π times the fluxes in the axial and radial direction over the line segments C that are normal to the fluxes and multiplied by $\bar{y}$.

Discretized Equations

The discretized axisymmetric flow equations for a single cell can be expressed as

$$\frac{\partial Q}{\partial t}V + \sum_{i=1}^{4} G_i \cdot B_i = HV \tag{19}$$

where i represents the four surfaces of the control volume. According to the foregoing discussions, Eq. (19) can be rewritten as

$$\left(\frac{\partial Q'}{\partial t} - H'\right) A + \sum_{i=1}^{4} [(E - E_v)'_i C_x + (F - F_v)'_i C_y] = 0 \tag{20}$$

where

$$\begin{aligned} Q' &= \hat{y}Q \\ H' &= \hat{y}H \\ (E - E_v)' &= \bar{y}(E - E_v) \\ (F - F_v)' &= \bar{y}(F - F_v) \end{aligned} \tag{21}$$

It is clear that A and C are the cross-sectional AREA and the boundary LENGTH, respectively, of a control volume in a two-dimensional, curvilinear coordinate system.

The above discussion shows that the discretized flow equations in a two-dimensional, curvilinear coordinate system can be easily transformed to the axisymmetric system with the following steps:

1) Multiply the time marching term by the radial coordinate of the control volume centroid $\hat{y}$.

2) Reformulate the divergence of the velocity vector according to Eq. (9).

3) Multiply the flux terms with the radial coordinate of the boundary centroid $\bar{y}$.

4) Add the source term [due to the axisymmetric coordinate, Eq. (4)], multiplied by the radial coordinate of the control volume centroid $\hat{y}$ to the radial momentum equation.

Concluding Remarks

The axisymmetric flow equations have been organized in a vector form suitable for CFD applications. The physical meaning of the source term in the radial momentum equation due to the axisymmetric coordinate system is discussed. The details of the volumetric and surface integrations for a single cell are examined. A convenient method of modifying existing two-dimensional codes to axisymmetric ones is proposed. Most importantly, no numerical accuracy has been sacrificed in calculating the volume and surface areas of the control volume with this method.

Reference

[1]Taylor, A. E., and Mann, W. R., *Advanced Calculus*, 2nd ed., Wiley, New York, 1972, p. 486.